On harmonic morphisms from 4-manifolds to Riemann surfaces and local almost Hermitian structures

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Abstract

We investigate the structure of a harmonic morphism F from a Riemannian 4-manifold M^4 to a 2-surface N^2 near a critical point m_0. If m_0 is an isolated critical point or if M^4 is compact without boundary, we show that F is pseudo-holomorphic w.r.t. an almost Hermitian structure defined in a neighbourhood of m_0.
If M^4 is compact without boundary, the singular fibres of F are branched minimal surfaces.

1 Introduction

1.1 Background

A harmonic morphism $F : M \longrightarrow N$ between two Riemannian manifolds (M, g) and (N, g) is a map which pulls back local harmonic functions on N to local harmonic functions on M. Although harmonic morphisms can be traced back to Jacobi, their study in modern times was initiated by Fuglede and Ishihara who characterized them using the notion of horizontal weak conformality, or semiconformality:

Definition 1. *(see [B-W] p.46) Let $F : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds and let $x \in M$. Then F is called horizontally weakly conformal at x if either*
1) $dF_x = 0$

2) dF_x maps the space $Ker(dF_x)^\perp$ conformally onto $T_{F(x)}N$, i.e. there exists a number $\lambda(x)$ called the dilation of F at x such that

$$\forall X, Y \in Ker(dF_x)^\perp, h(dF_x(X), dF_x(X)) = \lambda^2(x) g(X, Y).$$

The space $Ker(dF_x)$ (resp. $Ker(dF_x)^\perp$) is called the vertical (resp. horizontal) space at x.

Fuglede and Ishihara proved independently

Theorem 1. *([Fu],[Is]) Let $F : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. The following two statements are equivalent:*
1) For every harmonic function $f : V \longrightarrow \mathbb{R}$ defined on an open set V of N, the function $f \circ F$ defined on the open set $F^{-1}(V)$ of M is harmonic.
2) The map F is harmonic and horizontally weakly conformal.
Such a map is called a harmonic morphism.

When the target is 2-dimensional, Baird and Eells proved

Theorem 2. *([B-E]) Let $F : (M^m, g) \longrightarrow (N^2, h)$ be a smooth nonconstant horizontally weakly conformal map between a Riemannian manifold (M^m, g) and a Riemannian 2-surface (N^2, h). Then F is harmonic (hence a harmonic morphism) if and only if the fibres of F at regular points are minimal submanifolds of M.*

It follows from Th.2 that holomorphic maps from a Kähler manifold to a Riemann surface are harmonic morphisms; this raises the question of the interaction between harmonic morphisms to surfaces and holomorphic maps. John Wood studied harmonics morphisms $F : M^4 \longrightarrow N^2$ from an Einstein 4-manifold M^4 to a Riemann surface N^2 and exhibited an *integrable* Hermitian structure J on the regular points of F w.r.t. which F is holomorphic ([Wo]). He extended J to some of the critical points of F and the second author extended it to all critical points ([Vi1]).
By contrast, Burel constructed many harmonic morphisms from $\mathbb{S}^4$ to $\mathbb{S}^2$, for non-canonical metrics on $\mathbb{S}^4$ ([Bu]); he was building upon previous constructions on product of spheres by Baird and Ou ([B-O]). Yet it is well-known that $\mathbb{S}^4$ does not admit any global almost complex structure (see for example [St] p.217).

1.2 The results

In the present paper, we continue along the lines of [Wo] and [Vi1] and investigate the case of a harmonic morphism $F : M^4 \longrightarrow N^2$ from a general Riemannian 4-manifold M^4 to a 2-surface N^2. In [Wo] the integrability of J follows from the Einstein condition so we cannot expect to derive an integrable Hermitian structure in the general case. Could F be pseudo-holomorphic w.r.t. some *almost* Hermitian structure J on M^4? Burel's example on $\mathbb{S}^4$ tells us that we cannot in general expect J to be defined on all of M^4: the most we can expect is for F to be pseudo-holomorphic w.r.t. a local almost Hermitian structure. We feel that this should be true in general; however, we only are able to prove it in two cases:

Theorem 3. *Let (M^4, g) be a Riemannian 4-manifold, let (N^2, h) be a Riemannian 2-surface and let $F : M^4 \longrightarrow N^2$ be a harmonic morphism. Consider a critical point m_0 in M^4 and assume that one of the following assertions is true*
1) m_0 is an isolated critical point of F
OR
2) (M^4, g) is compact without boundary (and m_0 need not be isolated).
Then there exists an almost Hermitian structure J in a neighbourhood of m_0 w.r.t. which F is pseudo-holomorphic.

NB. The pseudo-holomorphicity of F means: if $m \in M^4$ and $X \in T_m M^4$,

$$dF(JX) = j \circ dF(X)$$

where j denotes the complex structure on N^2.

We can use the work of [McD] and [M-W] on pseudo-holomorphic curves: under the assumptions of Th. 3, the local topology of a singularity of a fibre of F is the same as the local topology of a singular complex curve in $\mathbb{C}^2$.

We derive from the proof of Th. 3

Corollary 1. *Let $F : M^4 \longrightarrow N^2$ be a harmonic morphism from a compact Riemannian 4-manifold without boundary to a Riemann surface and let u_0 be a singular value of F. Then the preimage $F^{-1}(u_0)$ is a (possibly branched) minimal surface.*

If the manifold M^4 is Einstein, the Hermitian structure constructed by Wood is parallel on the fibres of the harmonic morphism and has a fixed orientation. In the general case, around regular points of F, there are two local almost Hermitian structures making F pseudo-holomorphic; they have opposite orientations and we denote them J_+ and J_-. We follow Wood's computation without assuming M^4 to be Einstein and get a a bound on the product of the $\|\nabla J_\pm\|$'s (we had hoped for a local bound on one of the $\|\nabla J_\pm\|$'s):

Proposition 1. *Let (M^4, g) be a Riemannian 4-manifold, let (N^2, h) be a Riemannian 2-surface and let $F : M^4 \longrightarrow N^2$ be a harmonic morphism. We denote by j the complex structure on N^2 compatible with the metric and orientation. For a regular point m of F, we let J_+ (resp. J_-) be the almost complex structure on $T_m M^4$ such that*
i) J_+ and J_- preserve the metric g
ii) J_+ (resp. J_-) preserves (resp. reverses) the orientation on $T_m M^4$
iii) the map $dF : (T_m M^4, J_\pm) \longrightarrow (N^2, j)$ is complex-linear.
Let K be a compact subset of M^4: there exists a constant A such that, for every regular point m of M^4 in K and every unit vertical tangent vector T at m,

$$\|\nabla_T J_+\| \|\nabla_T J_-\| \leq A$$

when ∇ denotes the connection induced by the Levi-Civita connection on M^4.

1.3 Sketch of the paper

In §2, we recall that the lowest order term of the Taylor development at a critical point of F is a homogeneous holomorphic polynomial; we use it to control one of the two local pseudo-Hermitian structures for which F is pseudo-holomorphic at regular points close to m_0. We express this in the Main Lemma (§2.3) and Th. 3 1) follows almost immediately (§3). In §4 we prove Th. 3 2) using the twistor constructions of Eells and Salamon ([Ee-Sal]): the twistor space $Z(M^4)$ is a 2-sphere bundle above M^4 endowed with an almost complex structure $\mathcal{J}$ and the regular fibres of F lift to $\mathcal{J}$-holomorphic curves in $Z(M^4)$. The assumptions of Th. 3 2) enable us to prove that these curves have bounded area so we can use Gromov's compactity theorem: as we approach m_0, the lifts of the regular fibres of F in each of the two twistor spaces of M^4 converge to a $\mathcal{J}$-holomorphic curve. The Main Lemma enables us to pick one of the two orientations so that the limit curve

has no vertical component near m_0: near m_0, it is the lift of the fibre of F containing m_0. This is the key point in the proof of Th. 3 2).
In §5, we prove Prop 1 using an identity which Wood established to prove the superminimality of the fibres in the Einstein case.

For background and detailed information about harmonic morphisms, we refer the reader to [B-W].

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2 The main lemma

2.1 The almost complex structure at regular points

A REMARK ABOUT THE NOTATION. If m is a point in M^4, we denote by $|m|$ the distance of m to m_0. We introduce several constants, which we number C_1,...,C_{10},...; they all have the same goal which is to say that one quantity or another is a $\mathcal{O}(|m|)$, so the reader in a hurry can ignore the indices and think of a single constant C.

Let m be a regular point of F in M^4; as we mentioned above in Def. 1, the tangent space of M^4 at a regular point m of F splits as follows:

$$T_m M^4 = V_m \oplus H_m \tag{1}$$

where the vertical space V_m is the space tangent at m to the fibre $F^{-1}(F(m))$ and the horizontal space H_m is the orthogonal complement of V_m in $T_m M^4$.

2.2 The symbol and its extension in a neighbourhood of a critical point

We use the notations of Th.3 and we let m_0 be a critical point of F. We denote by k, $k > 1$, the order of F at m_0; namely, if (x_i) is a coordinate

system centered at m_0, m_0 being identified with $(0,...,0)$, we have
1) for every multi-index $I = \{i_1, ..., i_4\}$ with $|I| \le k-1$,

$$\frac{\partial^{|I|} F}{\partial^{i_1} x_1 ... \partial^{i_4} x_4}(0,...,0) = 0$$

2) there exists a multi-index $J = \{j_1, ..., j_4\}$ with $|J| = k$ such that

$$\frac{\partial^{k} F}{\partial^{j_1} x_1 ... \partial^{j_4} x_4}(0,...,0) \neq 0$$

The lowest order term of the Taylor development of F at m_0 is a homogeneous polynomial

$$P_0 : T_{m_0} M^4 \longrightarrow T_{F(m_0)} N^2$$

of degree k called the *symbol* of F at m_0. Fuglede showed ([Fu]) that P_0 is a harmonic morphism between $T_{m_0}M^4$ and $T_{F(m_0)}N^2$; it follows from [Wo] that P is a holomorphic polynomial of degree k for some orthogonal complex structure J_0 on $T_{m_0}M^4$.
REMARK. The complex structure J_0 is not always uniquely defined as the following two examples illustrate:
1) $P_0(z_1, z_2) = z_1 z_2$: J_0 is uniquely defined
2) $P_0(z_1, z_2) = z_1^2$: there are two possible J_0's with opposite orientations.

2.3 The main lemma

We identify a neighbourhood U of m_0 with a ball in $\mathbb{R}^4$, the point m_0 being identified with the origin and we let (x_i) a system of normal coordinates in U. We pick these coordinates so that, at the point m_0, we have

$$J_0 \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} \qquad J_0 \frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4} \tag{2}$$

We extend J_0 in U by requiring (2) to be verified for all points in U. Of course J_0 does not necessarily preserve the metric outside of m_0, nevertheless there exists a constant C_2 such that, for a vector X tangent at a point m

$$| < J_0 X, J_0 X > - < X, X > | \le C_2 |m|^2 \|X\|^2 \quad | < J_0 X, X > | \le C_2 |m|^2 \|X\|^2 \tag{3}$$

We identify a neighbourhood of $F(m_0)$ with a disk in $\mathbb{C}$ centered at the origin, with $F(m_0)$ identified with 0. We also extend P_0 in U by setting

$$P : U \longrightarrow \mathbb{C}$$

$$P(x_1, ..., x_4) = P_0(x_1 + ix_2, x_3 + ix_4).$$

It is clear that for $m \in U$ and $i = 1, ..., 4$

$$\frac{\partial P}{\partial x_i}(m) = \frac{\partial P_0}{\partial x_i}(x(m))$$

hence P is J_0-holomorphic.

Main Lemma. *Let M^4 be a Riemannian 4-manifold, N^2 a Riemannian 2-surface and $F : M^4 \longrightarrow N^2$ a harmonic morphism. We consider a critical point m_0 of F which we do not assume isolated. We denote by P_0 the symbol of F at m_0, assumed to be holomorphic for a parallel Hermitian complex structure J_0 on $T_{m_0}M^4$ and we extend J_0 to a neighbourhood of m_0 as explained above.*
In a neighbourhood U of m_0, there exists an almost Hermitian structure J continuously defined on the regular points of F in U such that
1) J has the same orientation as J_0
2) F is pseudo-holomorphic w.r.t. J.
Moreover, for a point m in U

$$|J(m) - J_0(m)| \leq C_3|m|$$

for some positive constant C_3 independent of m.

Proof. We let $\Psi = F - P$. By definition of the symbol, there exist C_4, C_5 such that

$$\forall m \in U, \forall X \in T_mM |\Psi(m)| \leq C_4|m|^{k+1}, \quad |d\Psi(m)(X)| \leq C_5|m|^k\|X\| \quad (4)$$

We let (ϵ_1, ϵ_2) be a local positive orthonormal basis of N^2 in a neighbourhood of u_0. Denoting by j the complex structure on N, we have

$$\epsilon_2 = j\epsilon_1 \quad (5)$$

If m is a regular point of F, we define two unit orthogonal vectors e_1, e_2 in H_m such that

$$dF(e_1) = \lambda(m)\epsilon_1 \quad dF(e_2) = \lambda(m)\epsilon_2 \quad (6)$$

where $\lambda(m)$ denotes the dilation of F at m (see Def.1 and [B-W] pp. 46-47). Next we pick an orthonormal basis (e_3, e_4) of H_m in a way that (e_1, e_2, e_3, e_4) is of the orientation defined by J_0. We define the almost complex structure J by setting

$$Je_1 = e_2 \qquad Je_3 = e_4 \tag{7}$$

We first show that $J_0 e_1$ is close to e_2; we set

$$J_0 e_1 = ae_1 + be_2 + v \tag{8}$$

where $a, b \in \mathbb{R}$ and $v \in V_m$.
Since $a =< J_0 e_1, e_1 >$, we get from (3)

$$|a| \leq C_2 |m|^2 \tag{9}$$

Next we compute $dF(J_0 e_1)$:

$$\begin{aligned} dF(J_0 e_1) &= dP(J_0 e_1) + d\Psi(J_0 e_1) = jdP(e_1) + d\Psi(J_0 e_1) \\ &= jdF(e_1) - jd\Psi(e_1) + d\Psi(J_0 e_1) \end{aligned} \tag{10}$$

On the other hand, it follows from (8) that

$$\begin{aligned} dF(J_0 e_1) &= adF(e_1) + bdF(e_2) \\ &= adF(e_1) + jbdF(e_1) \end{aligned} \tag{11}$$

by definition of e_1 and e_2 (see (6)).
Putting (10) and (11) together, we get

$$j(1-b)dF(e_1) = adF(e_1) + jd\Psi(e_1) - d\Psi(J_0 e_1)$$

and using (6), we derive

$$|1-b|\lambda(m) = |adF(e_1) + jd\Psi(e_1) - d\Psi(J_0 e_1)| \tag{12}$$

We already know that the right-hand side of (12) is a $\mathcal{O}(|m|^k)$; in order to show that $|1-b|$ is a $\mathcal{O}(|m|)$, we need to bound $\lambda(m)$ below.

Lemma 1. *There exists a $C_6 > 0$ such that, for m small enough,*

$$\lambda(m) \geq C_6 |m|^{k-1}$$

Proof. First we notice that

$$\lambda(m) = \sup_{X \in T_m M^4, \|X\|=1} \|dF(m)X\| \tag{13}$$

Indeed, take a vector $X \in T_m M$ with $\|X\| = 1$. We split it into $X = X_v + X_h$ with X_v vertical and X_h horizontal. Then $\|X_v\|^2 + \|X_h\|^2 = 1$ and

$$\|dF(m)X\| = \|dF(m)X_h\| = \lambda(m)\|X_h\| \leq \lambda(m).$$

Since P_0 is of degree k, there exists C_7 such that for m small enough

$$\sup_{X \in T_m M^4, \|X\|=1} \|dP(m)X\| \geq C_7|m|^{k-1}.$$

It follows that, for $m \in U$ and $X \in T_m M^4$ with $\|X\| = 1$, we have

$$|dF(m)X| = |dP(m)X + d\Psi(m)X| \geq |dP(m)X| - |d\Psi(m)X|$$

$$\geq C_7|m|^{k-1} - C_5|m|^k$$

We take m small enough so that $C_5|m| \leq \frac{C_7}{2}$ and the lemma follows by taking $C_6 = \frac{C_7}{2}$. □

It follows from (12) and from Lemma 1 that

$$|b-1| \leq C_9|m| \tag{14}$$

for m small enough and some constant C_9.
To estimate $\|v\|$, we use (3) to write for m small enough

$$|\|J_0 e_1\|^2 - 1| = |a^2 + b^2 + \|v\|^2 - 1| \leq C_2|m|^2 \tag{15}$$

Hence

$$\|v\|^2 \leq C_2|m|^2 + a^2 + |b^2 - 1|$$

and it follows from (9) and (14) that

$$\|v\| \leq C_{11}|m| \tag{16}$$

for some positive constant C_{11}.
We can now conclude. Since

$$\|Je_1 - J_0 e_1\| = \|e_2 - J_0 e_1\| \leq |a| + |b-1| + \|v\|$$

$\|Je_1 - J_0e_1\|$ is a $\mathcal{O}(|m|)$; similarly for $\|Je_2 - J_0e_2\|$.

We now prove that $\|Je_3 - J_0e_3\|$ is a $\mathcal{O}(|m|)$: there are no new ideas so we skip the details. We write

$$J_0e_3 = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4$$

Since (e_i) is an orthonormal basis,

$$|\alpha| = |< J_0e_3, e_1 >| \leq |< J_0e_1, e_3 >| + C_2|m|^2$$

using (3); it follows from the estimates above for J_0e_1 that α (and for the same reason β) is a $\mathcal{O}(|m|)$.
We also derive from (3) that

$$|\gamma| = |< J_0e_3, e_3 >| \leq C_2|m|^2$$

Now that we know that α, β and γ are $\mathcal{O}(|m|^2)$'s, we focus on δ and derive from (3)

$$|\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - 1| = |\|J_0e_3\|^2 - 1| \leq C_2|m|^2$$

It follows that $|\delta^2 - 1|$ is an $\mathcal{O}(|m|^2)$, hence δ is either close to 1 or to -1: let us prove that δ is positive, using orientation arguments.
In a neighbourhood of m, we identify $\Lambda^4(M)$ with $\mathbb{R}$ so we can talk of signs of 4-vectors. If we denote by $\star$ the Hodge star operator, the sign of $e_1 \wedge J_0e_1 \wedge \star(e_1 \wedge J_0e_1)$ gives us the orientation of J_0 hence, by our assumption, it is of the same sign as $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We have seen that, close to m, J_0e_1 is close to e_2, hence $e_1 \wedge J_0e_1 \wedge \star(e_1 \wedge J_0e_1)$ and $e_1 \wedge J_0e_1 \wedge e_3 \wedge J_0e_3$ have the same sign; this latter 4-vector has the same sign as $\delta e_1 \wedge e_2 \wedge e_3 \wedge e_4$. It follows that δ is positive.

Hence $\|J_0e_3 - Je_3\|$ is a $\mathcal{O}(|m|)$ and so is $\|J_0e_4 - Je_4\|$, by identical arguments; this concludes the proof of the Main Lemma. $\square$

3 Proof of Th.3 1): an isolated critical point

If m_0 is an isolated critical point, the almost complex structure J given by the Main Lemma is defined in $U \backslash \{m_0\}$ where U is a neighbourhood of m_0. At the point m_0, we put $J(m_0) = J_0$ and the Main Lemma tells us that the resulting almost complex structure is continuous.

4 Proof of Th.3 2):M is compact without boundary

4.1 Background: twistor spaces

We give here a brief sketch of Eells-Salamon's work with twistors ([Ee-Sal]); the reader can find a more detailed exposition in Chap. 7 of [B-W].
The *twistor space* $Z^+(M^4)$ (resp. $Z^-(M^4)$) of an oriented Riemannian 4-manifold (M^4, g) is the 2-sphere bundle defined as follows: a point in $Z^+(M^4)$ (resp. $Z^-(M^4)$) is of the form (J_0, m_0) where m_0 is a point in M^4 and J_0 is an orthogonal complex structure on $T_{m_0}M^4$ which preserves (resp. reverses) the orientation on $T_{m_0}M^4$. The twistor spaces $Z^\pm(M^4)$ admit the following almost complex structures $\mathcal{J}_\pm$.
We split the tangent space $T_{(J_0,m_0)}Z^\pm(M^4)$ into a horizontal space $\mathcal{H}_{(J_0,m_0)}$ and a vertical space $\mathcal{V}_{(J_0,m_0)}$. Since $\mathcal{H}_{(J_0,m_0)}$ is naturally identified with $T_{m_0}M^4$, we define $\mathcal{J}_\pm$ on $\mathcal{H}_{(J_0,m_0)}$ as the pull-back of J_0 from $T_{m_0}M^4$; the fibre above m_0 is an oriented 2-sphere so we define $\mathcal{J}_\pm$ on $\mathcal{V}_{(J_0,m_0)}$ as the opposite of the canonical complex structure on this 2-sphere. If S is an oriented 2-surface in M^4, it has a natural lift inside the twistor spaces: a point p in S lifts to the point (J_p, p) in $Z^+(M^4)$ (resp. $Z^-(M^4)$), where J_p is the orthogonal complex structure on T_pM^4 which preserves (resp. reverses) the orientation and for which the oriented plane T_pS is an oriented complex line.
Jim Eells and Simon Salamon proved

Theorem 4. *([Ee-Sal]) Let (M^4, g) be a Riemannian 4-manifold. A minimal surface in M^4 lifts into a $\mathcal{J}_+$-holomorphic (resp. $\mathcal{J}_-$-holomorphic) curve in $Z^+(M^4)$ (resp. $Z^-(M^4)$). Conversely, every non vertical $\mathcal{J}_\pm$-holomorphic curve in $Z^\pm(M^4)$ is the lift of a minimal surface in M^4.*

4.2 Convergence of the twistor lifts of regular fibres

We assume M^4 to be oriented: Th.3 is local so if M^4 is not oriented, we endow a ball centered in at m_0 with an. We endow M^4 with the orientation given by the complex structure J_0 on $T_{m_0}M^4$ defined by the symbol (see §2.2). From now on we drop the superscript $+$ and we write $Z(M^4)$ for $Z^+(M^4)$.
We denote $u_0 = F(m_0)$ and we let (u_n) be a sequence of regular values of F which converges to u_0. The preimages of the u_n's are smooth compact closed

2-submanifolds of M^4. For every positive integer n, we let

$$S_n = F^{-1}(u_n).$$

Lemma 2. *The S_n's all have the same area.*

Proof. The singular values of F are discrete so we can assume that the S_n's are all deformation of one another; moreover they are all minimal. It follows from the formula for the first variation of area that if (Σ_t), $t \in [0,1]$ is a family of minimal surfaces without boundary in a compact manifold, $\frac{d}{dt}area(\Sigma_t) = 0$, hence $area(\Sigma_t)$ is constant, for $t \in [0,1]$. $\square$

We denote by $\tilde{S}_n$ the lift of S_n into $Z^+(M^4)$: Th. 4 tells us that they are $\mathcal{J}$-holomorphic curves. Moreover we have

Lemma 3. *There exists a constant C such that, for every positive integer n,*

$$area(\tilde{S}_n) \leq C.$$

Proof. We parametrize the $\tilde{S}_n$'s by maps

$$\gamma_n : S_n \longrightarrow \tilde{S}_n$$

We let (e_1, e_2) be an orthonormal basis of the tangent bundle TS_n and we denote by $\tilde{e}_1, \tilde{e}_2$ their lift in $Z(M^4)$. For $i = 1, 2$, we split $\tilde{e}_i$ into vertical and horizontal components,

$$\tilde{e}_i = \tilde{e}_i^h + \tilde{e}_i^v$$

We write the area element of $\tilde{S}_n$:

$$\|\tilde{e}_1 \wedge \tilde{e}_2\| \leq \|\tilde{e}_1^h \wedge \tilde{e}_2^h\| + \|\tilde{e}_1^h \wedge \tilde{e}_2^v\| + \|\tilde{e}_1^v \wedge \tilde{e}_2^h\| + \|\tilde{e}_1^v \wedge \tilde{e}_2^v\| \tag{17}$$

Integrating (17) and using Cauchy-Schwarz inequality, we get

$$area(\tilde{S}_n) \leq area(S_n) + 2\sqrt{area(S_n)}\sqrt{\int_{S_n} \|\nabla\gamma_n\|^2} + \int_{S_n} \|\nabla\gamma_n\|^2$$

where ∇ denote the connection on $Z(M^4)$ induced by the Levi-Civita connection on M^4.

Lemma 4. *There exists a constant A such that, for every positive n*

$$\int_{S_n} \|\nabla\gamma_n\|^2 \leq A$$

Proof. We need to introduce a few notations to give a formula for the integral in Lemma 4. For every n, we let NS_n be the normal bundle of S_n in M^4 and we endow it with a local orthonormal basis (e_3, e_4). We denote by R the curvature tensor of (M^4, g) and we put

$$\Omega^T =< R(e_1, e_2)e_1, e_2 > \qquad \Omega^N =< R(e_1, e_2)e_3, e_4 >$$

Finally we let $c_1(NS_n)$ be the degree of NS_n i.e. the integral of its 1st Chern class; it changes sign with the orientation of M^4. Note that other authors (e.g. [C-T]) denote it by $\chi(NS_n)$, by analogy with the Euler characteristic. We denote by dA the area element of S_n and we derive from [C-T] (see also [Vi 2])

$$\frac{1}{2}\int_{S_n} \|\nabla\gamma_n\|^2 = -\chi(S_n) - c_1(NS_n) + \int_{S_n} \Omega^T dA + \int_{S_n} \Omega^N dA \tag{18}$$

The critical values of F are isolated, hence the regular fibres all have the same homotopy type and the same homology class $[S_n]$ in $H_2(M^4, \mathbb{Z})$. In particular, $|\chi(S_n)|$ does not depend on n. The S_n's are embedded hence $c_1(NS_n)$ is equal to the self-intersection number $[S_n].[S_n]$ which does not depend on n either.
Since M^4 is compact, the expression $| < R(u, v)w, t > |$ has an upper bound for all the 4-uples of unit vectors (u, v, w, t). It follows that the integrals in Ω^T and Ω^N in (18) have a common bound in absolute value.
In conclusion, all the terms in the RHS of (18) are bounded in absolute value uniformely in n. □

Lemma 3 follows immediately.

□

Thus the $\tilde{S}_n$'s are $\mathcal{J}$-holomorphic curves of bounded area in $Z(M^4)$: Gromov's result ([Gro]) ensures that they admit a subsequence which converge in the sense of cusp-curves to a $\mathcal{J}$-holomorphic curve C.

Lemma 5. *We denote by $\pi : Z(M^4) \longrightarrow M^4$ the natural projection. Then*

$$\pi(C) = F^{-1}(u_0).$$

Proof. The map $\pi \circ F$ is continuous, so it is clear that $\pi(C) \subset F^{-1}(u_0)$.
To prove the reverse inclusion, we take a point $p \in F^{-1}(u_0)$ and we claim

Claim 1. *There exists a subsequence* $(u_{s(n)})$ *of* (u_n) *and a sequence of points* (p_n) *of* M^4 *converging to* p *with*

$$F(p_n) = u_{s(n)}$$

for every positive integer n.

Indeed, if Claim 1 was not true, we would have the following

Claim 2. $\exists \epsilon > 0$ *such that* $\forall n \in \mathbb{N}^*$ *and* $\forall m \in M^4$ *with* $F(m) = u_n$, *we have*

$$d(m, p) > \epsilon.$$

If Claim 2 was true, the set $F(B(x, \epsilon))$ would contain u_0 but would not be a neighbourhood of u_0, a contradiction of the fact that a harmonic morphism is open ([Fu],[B-W] p.112).
So Claim 1 is true: if we denote by (J_n, p_n) the pullback of the p_n's in the twistor lifts $\tilde{S}_n$, they admit a subsequence which converges to a point $(\hat{J}, p)$, for some $\hat{J}$ in the twistor fibre above p. Clearly $(\hat{J}, p)$ belongs to C, hence p belongs to $\pi(C)$ and Lemma 5 is proved. $\square$

Their are a finite number of points $p_1, ..., p_k \in M^4$ and positive integers $q_1, ..., q_k$ such that the curve C can be written

$$C = \Gamma + \sum_{i=1}^{k} q_i Z_{p_i} \tag{19}$$

where Γ is a $\mathcal{J}$-holomorphic curve with no vertical components and the Z_{p_i}'s are the twistor fibres above the p_i's. It follows from Lemma 5 that

$$\pi(\Gamma) = F^{-1}(u_0).$$

We derive that $F^{-1}(u_0)$ is a minimal surface possibly with branched points and having Γ as its twistor lift.
Note that the presence of twistor fibres in (19) is to be expected: when a sequence of smooth minimal surfaces converges to a minimal surface with singularities, its twistor lifts can experience bubbling off of twistor fibres above singular points (see [Vi2] for a more detailed discussion of this phenomenon). However, in the present case, the Main Lemma excludes such bubbling-off in a neighbourhood of m_0:

Lemma 6. *There exists an* $\epsilon > 0$ *such that, if* q_i *is one of the points appearing in (19),*

$$dist(m_0, p_i) > \epsilon$$

Proof. Since the q_i's are finite in number, it is enough to prove that m_0 is not one of them.
The almost complex structure J_0 appearing in the Main Lemma does not necessarily preserve the metric outside of m_0; so we introduce the bundle $\mathcal{C}$ of *all* the complex structures on TM^4 which preserve the orientation. It contains the bundle $Z(M^4)$ and embeds into the bundle $GL(TM^4)$. We denote by $d_{\mathcal{C}}$ the distance on $\mathcal{C}$ induced by the metric on $GL(TM^4)$ and by d_{M^4} the distance in M^4.

Lemma 7. $\forall\epsilon > 0 \quad \exists\eta > 0$ *such that*

$$d_{M^4}(m, m_0) < \eta \Rightarrow d_{\mathcal{C}}[(J(m), m), (J_0, m_0)] < \epsilon$$

Proof. $d_{\mathcal{C}}[(J(m), m), (J_0, m_0)]$

$$\leq d_{\mathcal{C}}[(J(m), m), (J_0(m), m)] + d_{\mathcal{C}}[(J_0(m), m), (J_0, m_0)] \tag{20}$$

We bound the first term in (20) using the Main Lemma; the second term is bounded because $J_0 : U \longrightarrow \mathcal{C}$ is continuous. □

If m is a regular point of F, we denote by $\gamma(m)$ the point above m in the twistor lift of $F^{-1}(F(m))$; in the Main Lemma, we defined the almost complex structure $J(m)$. The tangent plane to the fibre at m is a complex line for both $\gamma(m)$ and $J(m)$; since $\gamma(m)$ and $J(m)$ both preserve the orientation, it follows that $\gamma(m) = \pm J(m)$. We can get rid of the $\pm$ by saying that $F^{-1}(u_0)$ is a 2-dimensional CW-complex, hence $B(m_0, \epsilon)\backslash F^{-1}(u_0)$ is connected: there is a $s \in \{-1, +1\}$ such that for every regular point m of F near m_0,

$$\gamma(m) = sJ(m) \tag{21}$$

We rewrite Lemma 7: $\forall\epsilon > 0 \quad \exists\eta > 0$ such that for a regular point m,

$$d_M(m, m_0) < \eta \Rightarrow d_{\mathcal{C}}[(\gamma(m), m), (sJ_0, m_0)] < \epsilon \tag{22}$$

If the whole twistor fibre $Z_{m_0}M^4$ was included in C, it would be in the closure of the union of the twistor lifts of the regular fibres of F in a neighbourhood of m_0: we see from (22) that this is impossible. This concludes the proof of Lemma 6. □

4.3 Construction of the almost complex structure

We now construct a local section of $Z(M^4)$, for which F holomorphic. As in [Vi1], we work first on the space $\mathbb{P}(Z(M^4))$ obtained by taking the quotient of each twistor fibre by its antipody; if J is an element of $Z(M^4)$, we denote by $\bar{J}$ its image in $\mathbb{P}(Z(M^4))$.
If m is a regular point of F, there are 2 complex structures, J_1 and J_2, on $T_{m_0}M^4$ for which the unoriented planes V_m and H_m are complex lines. These two complex structures verify $J_1 = -J_2$, hence they define the same point, denoted $\bar{J}(m)$, in $\mathbb{P}(Z_m(M^4))$. To extend this section of $\mathbb{P}(Z(M^4))$ above $F^{-1}(u_0)$, we state

Lemma 8. *There exists an $\epsilon > 0$ such that every $m \in B(m_0, \epsilon) \cap F^{-1}(u_0)$ has either a single preimage in Γ or exactly two antipodal preimages in Γ.*

Proof. We let ϵ be a number satisfying Lemma 6 and we pick $m \in B(m_0, \epsilon) \cap F^{-1}(u_0)$. Since Γ has no vertical component above $B(m, \epsilon) \cap F^{-1}(u_0)$, it meets $Z_m(M^4)$ at a discrete number of points. Let us assume that J_1 and J_2 are two different elements of $\Gamma \cap Z_m(M^4)$. There exist two non vertical possibly branched disks Δ_1 and Δ_2 in Γ containing (J_1, m) and (J_2, m) respectively. Each one of the two Δ_i's is the twistor lift of a possibly branched disk D_i of $F^{-1}(u_0)$. The disks D_1 and D_2 meet at m: if they have different tangent planes at m, this implies that m is a singular point of $F^{-1}(u_0)$. Since $F^{-1}(u_0)$ is a closed minimal surface, its singular points are discrete so we can make ϵ small enough so that there is not singular point in $F^{-1}(u_0) \cap B(m_0, \epsilon)$ except for possibly m_0.
So we assume that m_0 is a singular point of $F^{-1}(u_0)$. Because the symbol is J_0 holomorphic, all planes tangent to m_0 at $F^{-1}(u_0)$ are J_0-complex lines and it follows that $J_1 = -J_2 = \pm J_0$. □

We denote by $\bar{\Gamma}$ the projection of Γ in $\mathbb{P}(Z(M^4))$ and by $\bar{J}$ the local section of $\bar{\Gamma}$ given by Lemma 8.

Lemma 9. *There exists a small $\epsilon > 0$ such that the map*

$$B(m_0, \epsilon) \longrightarrow \mathbb{P}(Z(M^4))$$

$$m \mapsto \bar{J}(m)$$

is continuous.

Proof. Since $\bar{J}$ is continuous above $U\backslash F^{-1}(u_0)$, we consider a sequence of points (p_n) in M^4 converging to a p_0 with $F(p_0) = u_0$. It is enough to the consider two cases
1st case: all the $F(p_n)$'s are regular values
2nd case: for every n, $F(p_n) = u_0$.
If (p_n) is a general sequence, we extract subsequences of the form 1) or 2).
1st case - For every n, $v_n = F(p_n)$ is a regular value of F.
i) First assume that $u_n = v_n$ for every n. Since Γ is the limit of the twistor lifts of the $F^{-1}(u_n)$ the sequence $(\bar{J}(p_n), p_n)$ converges to a point $(\bar{K}, p_0)$ in $\bar{\Gamma}$; Lemma 8 ensures that $\bar{K} = \bar{J}(p_0)$.
ii) In the general case, the v_n's converge to u_0 so we can proceed with the v_n's as we did with the u_n's and derive that the twistor lifts of the $F^{-1}(v_n)$'s converge in the sense of Gromov to the twistor lift of $F^{-1}(u_0)$ and conclude as in i).
2nd case For every n, $F(p_n) = u_0$. We denote by $\bar{\pi}$ the natural projection from $\mathbb{P}(Z(M^4))$ to M^4. Lemma 8 ensures that $\bar{\pi}$ restricts to a continuous bijection from $\bar{\Gamma} \cap \bar{\pi}^{-1}(\bar{B}(m_0, \frac{\epsilon}{2}))$ to $F^{-1}(u_0) \cap \bar{B}(m_0, \frac{\epsilon}{2})$; since these spaces are compact and Hausdorff, a continuous bijection between them is a homeomorphism (see for example [Han] p. 45). It follows that, if the p_n's converge to p_0, their preimages in $\bar{\Gamma}$ converge to the preimage of p_0; in other words, the $\bar{J}(p_n)$'s converge to $\bar{J}(p_0)$. $\square$

We conclude as in [Vi1]. We lift $\bar{J}$ above the set of regular points by taking for J the one complex structure on T_mM which renders dF holomorphic at that point - this requirement defines it uniquely on the horizontal space H_m and since, the orientation of J is given, there is also a unique possibility for J on V_m. By the same argument as in [Vi1], this extends to the entire $B(m_0, \epsilon)$. This concludes the proof of Th.3

5 Proof of Prop. 1

We begin by reproducing part of Wood's arguments ([Wo]).
We let m be a regular point of F and we denote by V_m (resp. H_m) the vertical (resp. horizontal) space at m. We let S_0V_m be the set of symmetric trace-free holomorphisms of V_m and we define the Weingarten map

$$A : H_m \longrightarrow S_0V_m$$

$$X \mapsto (U \mapsto \nabla^V_U X)$$

where $\nabla^V_U X$ denotes the vertical projection of $\nabla_U X$.

At a regular point m, we denote by J_+ (resp. J_-) the Hermitian structure on $T_m M^4$ w.r.t. which $dF : T_m M^4 \longrightarrow T_{F(m)} N^2$ is $\mathbb{C}$-linear and which preserves (resp. reverses) the orientation on $T_m M^4$. If M^4 is Einstein, Wood proves in Prop. 3.2 that all horizontal vectors X verify

$$^tA \circ A(J_\pm X) = J(^tA \circ A)(X).$$

If M^4 is not Einstein, we follow his proof to derive the existence of C_{11} such that, for every unit horizontal vector X tangent to a regular point of F in K,

$$\|^tA \circ A(J_\pm X) - J_\pm(^tA \circ A)(X)\| \leq C_{11} \tag{23}$$

We now put $T = e_1$ and we complete it into an orthonormal basis (e_1, e_2) of V_m; we pick an orthonormal basis (e_3, e_4) of H_m such that the almost complex structures verify

$$e_2 = J_+ e_1 = -J_- e_1 \qquad e_4 = J_+ e_3 = J_- e_3 \tag{24}$$

We let E_1 and E_2 be the following elements of $S_0 V_m$ defined by their matrices in the base (e_1, e_2).

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We write the matrix of A in the bases (e_3, e_4) and (E_1, E_2)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$a = - < \nabla_{e_1} e_1, e_3 > \qquad b = - < \nabla_{e_1} e_1, e_4 > \tag{25}$$

$$c = - < \nabla_{e_1} e_2, e_3 > \qquad d = - < \nabla_{e_1} e_2, e_4 > \tag{26}$$

The homomorphisms J_+ and J_- coincide on the basis (e_3, e_4) (see (24)); we compute

$$(^tA \circ A)J_\pm - J_\pm(^tA \circ A) = \begin{pmatrix} 2(ab + cd) & b^2 + d^2 - (a^2 + c^2) \\ b^2 + d^2 - (a^2 + c^2) & -2(ab + cd) \end{pmatrix}$$

and we derive from (23)

$$|ab + cd| \leq C_{11} \qquad |b^2 + d^2 - (a^2 + c^2)| \leq C_{11} \tag{27}$$

We take J to be J_+ or J_- and we write the Euclidean norm

$$\|\nabla_{e_1} J\|^2 = \sum_{i,j=1,\ldots,4} < (\nabla_{e_1} J) e_i, e_j >^2 \tag{28}$$

$$= \sum_{i,j=1,\ldots,4} (< \nabla_{e_1}(Je_i), e_j > - < J\nabla_{e_1} e_i, e_j >)^2 \tag{29}$$

$$= \sum_{i,j=1,\ldots,4} (< \nabla_{e_1}(Je_i), e_j > + < \nabla_{e_1} e_i, Je_j >)^2 \tag{30}$$

It is enough to take e_i vertical and e_j horizontal in (30):

Lemma 10. $\|\nabla_{e_1} J\|^2 = 2\sum_{1\leq i\leq 2, 3\leq j\leq 4} (< \nabla_{e_1}(Je_i), e_j > + < \nabla_{e_1} e_i, Je_j >)^2$

Proof. If e_i and e_j are both horizontal or both vertical, Prop. 2.5.16 i) of [B-W] yields

$$< \nabla_{e_1}(Je_i), e_j > = < J\nabla_{e_1} e_i, e_j > \tag{31}$$

Note that Baird-Wood's Prop. 2.5.16 is about horizontal vectors, but its proof works identically for vertical vectors.
Assume now that e_i is horizontal and e_j is vertical:

$$< \nabla_{e_1}(Je_i), e_j > + < \nabla_{e_1} e_i, Je_j > = - < Je_i, \nabla_{e_1} e_j > - < e_i, \nabla_{e_1}(Je_j) > \tag{32}$$

Putting together (30), (31) and (32) completes the proof of Lemma 10. □

We use the values given for the $J_\pm$ in (24) to derive

$$\begin{aligned}\frac{1}{2}\|\nabla_{e_1} J_\pm\|^2 = &(\pm < \nabla_{e_1} e_2, e_3 > + < \nabla_{e_1} e_1, e_4 >)^2 \\ &+(\pm < \nabla_{e_1} e_2, e_4 > - < \nabla_{e_1} e_1, e_3 >)^2 \\ &+(\pm < \nabla_{e_1} e_1, e_3 > - < \nabla_{e_1} e_2, e_4 >)^2 \\ &+(\pm < \nabla_{e_1} e_1, e_4 > + < \nabla_{e_1} e_2, e_3 >)^2\end{aligned} \tag{33}$$

We rewrite (33) in terms of the coefficients a, b, c, d of the matrix A introduced above (see (25) and (26)); we get after a short computation

$$\|\nabla_{e_1} J_+\|^2 = 4[(a-d)^2 + (b+c)^2] = 4[a^2 + b^2 + c^2 + d^2 - 2(ad - bc)] \tag{34}$$

$$\|\nabla_{e_1} J_-\|^2 = 4[(a+d)^2 + (b-c)^2] = 4[a^2 + b^2 + c^2 + d^2 + 2(ad - bc)] \quad (35)$$

hence

$$\|\nabla_{e_1} J_+\|^2 \|\nabla_{e_1} J_-\|^2 = 16[(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2] \quad (36)$$

We now bound (36) using (27). To this effect we put

$$a = R_1 \cos\theta \qquad c = R_1 \sin\theta \qquad b = R_2 \cos\alpha \qquad d = R_2 \sin\alpha \quad (37)$$

and we rewrite (36) as

$$\frac{1}{16} \|\nabla_{e_1} J_+\|^2 \|\nabla_{e_1} J_-\|^2 = (R_1^2 + R_2^2)^2 - 4R_1^2 R_2^2 \sin^2(\theta - \alpha) \quad (38)$$

$$= (R_1^2 + R_2^2)^2 - 4R_1^2 R_2^2 + 4R_1^2 R_2^2 \cos^2(\theta - \alpha) \quad (39)$$

$$= (R_1^2 - R_2^2)^2 + 4R_1^2 R_2^2 \cos^2(\theta - \alpha) \quad (40)$$

We now rewrite (27) as

$$|R_1 R_2 \cos(\theta - \alpha)| \leq C_{11} \qquad |R_1^2 - R_2^2| \leq C_{11} \quad (41)$$

and this allows us to bound (40) and conclude the proof of of Prop. 1.
□

References

[1] [B-E] P. Baird, J. Eells, *A conservation law for harmonic maps*, Geometry Symposium Utrecht 1980, Lecture Notes in Mathematics **894**, 1-25, Springer (1981)

[2] [B-O] P. Baird, Y.-L. Ou, *Harmonic maps and morphisms from multilinear norm-preserving mappings* Int. Jour. of Math. **8** (1997) 187-211

[3] [B-W] P. Baird, J.C. Wood, *Harmonic morphisms between Riemannian manifolds* London Math. Soc. Monographs **29**, Clarendon Press, Oxford (2003).

[4] [Bu] J.-M. Burel, *Applications et morphismes harmoniques dans une surface*, C.R.Acad.Sci. Paris, **332** Série I, (2001), 441-446.

[5] [C-T] J. Chen, G. Tian *Minimal surfaces in Riemannian 4-manifolds*, Geom. and Funct. Analysis, **7** (1997) 873-916.

[6] [Ee-Sal] J. Eells, S. Salamon*Twistorial constructions of harmonic maps of surfaces into four-manifolds*, Ann. Scuola Norm. Sup. di Pisa **12** (1985), 589-640.

[7] [Ee-Sam] J. Eells, J.H. Sampson, *Harmonic maps of Riemannian manifolds*, Amer. Jour. Math. **86** (1964), 109-160.

[8] [Fu] B. Fuglede *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier **28** 107-144.

[9] [Gro] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. math. **82**, (1985), 307-347.

[10] [Han] V.L. Hansen, *Fundamental concepts of modern analysis*, World Scientific (1999).

[11] [Is] T. Ishihara *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ. **19** (1979) 215-229.

[12] [M-W] M. Micallef, B. White *The structure of branch points in minimal surfaces and in pseudoholomorphic curves* Annals of Maths **139** (1994) 35-85.

[13] [McD] D. McDuff, *Singularities of J-holomorphic curves in almost complex 4-manifolds*, Jour. of Geom. Analysis **2** (1992), 249-266.

[14] [St] N. Steenrod, *The topology of fibre bundles* Princeton Landmarks in Math. and physics, PUP (1999; first published in 1951).

[15] [Vi1] M. Ville *Harmonic morphisms from Einstein 4-surfaces to Riemann surfaces*, Int. Jour. of Math. **14** (2003) 327-337.

[16] [Vi2] M. Ville *Milnor numbers for 2-surfaces in 4-manifolds* arXiv:math/0701896 (2007).

[17] [Wo] J.C. Wood, *Harmonic morphisms and Hermitian structures on Einstein 4-manifolds*, Int. Jour. of Math. **3** (1992) 415-439.

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